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Some New Aspects of Smooth Manifolds

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ABSTRACT: In the present paper some aspects of Classes Differentiability, Smooth Maps and their Differentials, Diffeomorphisms and Maps of Constant Rank, Smooth Submanifold and Local Flows are treated. Some important theorems, propositions, lemmas and examples are also given in this paper. A Lemma D2 is established which is related to smooth manifold.

***Keywords:** Classes Differentiability , Smooth Maps and their Differentials , Diffeomorphisms and Maps of Constant Rank ,Smooth Submanifold , Local Flows.*

1. INTRODUCTION

The emergence of differential geometry as a distinct discipline is generally credited to Carl Friedrich Gauss and Bernhard Riemann. Riemann first described manifolds in his famous habilitation lecture before the faculty at Gottingen. He motivated the idea of a manifold by an intuitive process of varying a given object in a new direction.The theory of smooth manifolds can be thought of a natural and very useful extension of the differ-ential calculus in $\mathbb{R}^n$ that its main theorems, which in our opinion are well represented by the inverse (or implicit) function theorem and the existence and uniqueness result for ordinary differ-ential equations, admit generalizations.In the present paper, we have been discussed Classes Differentiability, Smooth Maps and their Differentials, Diffeomorphisms and Maps of Constant Rank, Smooth Submanifold and Local Flows.

A. Classes Differentiability

Let$U \subseteq \mathbb{R}^n$ be an open subset. Let $x = (x^1, \dots\dots, x^n)$ denote the general point of U and let $p = (p^1, \dots\dots, p^n)$be a fixed but arbitrary point of U. Let $f: U \to \mathbb{R}$ be a function and let$L_p: U \to \mathbb{R}$ be an inhomogeneous linear (i.e. affine) map.

$$L_p(x) = c + \sum_{i=1}^{n} b_i x^i$$

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such that $L_p(p) = f(p)$.

Definition A1: If *f* andL_p are as above and if

$$lim_{x\to p}\frac{f(x) - L_p(x)}{\|x-p\|} = 0$$

Then L_p is called a derivative of *f* at *p*. If *f* admits a derivative at then *f* is said to be *differentiable* [7] at *p.*

Lemma A2: If

$$L_p(x) = c + \sum_{i=1}^{n} b_i x^i$$

is a derivative of *f* at p, then

$$b_i = \frac{\partial f}{\partial x^i}(p), \qquad 1 \le \text{i} \le \text{n}.$$

In particular, if *f* is differentiable *p* at these partial derivatives exist and the derivative $L_p$is unique.

Definition A3: The class of *continuous functions*$f: U \to \mathbb{R}$is denoted by $c^0(U)$.If $r \ge 1$,the clas $c^r(U)$ of functions $f: U \to \mathbb{R}$that are smooth of order *r* , $\frac{\partial f}{\partial x^i}$ exist and belong to $c^{r-1}(U), 1 \le \text{i} \le \text{n}$.The functions that are smooth of order *r* are also called c^r - smooth functions.

Definition A4: Let $c^\infty(U)$ be the set of *infinitely smooth functions* on *U* which is defined by

$$c^\infty(U) = \bigcap_{r\ge 0} c^r(U).$$

c^∞functions are usually called “smooth” .

Definition A5: Given ,$p \in U$, real valued functions *f* with *dom (f)* an subset of *U* and *p*∈*dom(f)*. The set of all c^1 curves s :$-(\partial, \varepsilon) \to U$such that *s(0) = p* is denoted by *S(U,P).* We say that s_1 and s_2 are *infinitesimally equivalent* at *p* if $s_{1,}s_2 \in S(U,P)$and we write $s_1 \approx_p s_2$ iff

$$\frac{d}{dt}f\left(s_1\Big|(t)\right)t = 0 = \frac{d}{dt}f\left(s_2\Big|(t)\right)t = 0, \forall f \in c^\infty(U,p);$$

where$c^\infty(U,p)$denote the set of *smooth.*

Definition A6: The *infinitesimal equivalence class* of s in $S(U,P)$is denoted by $s > \langle s \rangle_p$ and is called an *infinitesimal curve* at p. An infinitesimal curve at p is also called a *tangent vector* to U at p and the set $T_p(U) = S(U,P)/\approx_p$ of all tangent vectors at *p* is called the tangent space to U at p.

Lemma A7: Let$\langle s_1 \rangle_p, \langle s_2 \rangle_p \in T_p(U)$ and $a, b \in \mathbb{R}$. Then there is a unique infinitesimal $\langle s \rangle_p$curve such that the associated derivatives on $c^{\infty}(U,p)$satisfy

$D_{\langle s \rangle_p} = aD_{\langle s_1 \rangle_p} + bD_{\langle s_2 \rangle_p}$.

Proof. Here we will use the coordinates of $\mathbb{R}^{n}$ to prove this assertion. The important point is that the assertion itself is coordinate free. It is clear,

$$s(t) = as_1(t) + bs_2(t) - (a + b - 1)_p,$$

defined by coordinates operations for all values of *t* sufficiently near 0, is a c^1curve in *U* with s(0)= p and that this curve above represents the desired $\langle s \rangle_p \in T_p(U)$. By the above definition of infinitesimal equivalence, we say that $\langle s \rangle_p$ is completely determined by $D_{\langle s \rangle_p}$.

Definition A8: The elements $f, g \in c^{\infty}(U,p)$ are said to be *germinally equivalent* at *p* if there is an open neighborhood *W* of *p* in *U* such that $W \subseteq dom(f) \cap dom(g)$nd $f|W = g|W$ and we denoted by $f \sim_p g$.

Definition A9: The *germinal equivalenceclass*$[f]_p$of$f \in c^{\infty}(U,p)$of f is called the germ of *f* at *p*. The set $c^{\infty}(U,p)/\sim p$of germs at *p* is denoted by $\mathcal{B}_p$.

Definition A10:$e_p: \mathcal{B}_p \rightarrow \mathbb{R}$is the *evaluation map* which is defined by

$$e_p[f]_p = f(p).$$

Here $[f]_p$s the *germinal equivalence class*.

Lemma A11: The evaluation map $e_p: \mathcal{B}_p \rightarrow \mathbb{R}$ is a well defined homomorphism of algebras.

Definition A12: A derivative operator on$\mathcal{B}_p$ is an$\mathbb{R}$*linear mapping*$D: \mathcal{B}_p \to \mathbb{R}$ such that $D(ab) = D(a)e_p(b) + e_p(a)D(b), \forall a, b \in \mathcal{B}_p$. A derivative on $\mathcal{B}_p$ is also called a *tangent vector* to *U* at *p*. The set of all derivative on$\mathcal{B}_p$ is denoted by T_p or $T_p(U)$and is called the *tangent space* to *U* at *p*.

Lemma A13: If c is a constant function on *U* and *D*$\in T_p(U)$,then $D[C]_p = 0$.

Proof. First we consider the case c=1.Now we can write

$D[1]_p = D([1]_p[1]_p)$

$= D([1]_p e_p([1]_p) + e_p([1]_p) D[1]_p) = 2D[1]_p$.

From which it follows that $D[1]_p = 0$.Now for arbitrary constant *c*, we can write

$D[c]_p = cD[1]_p = 0$,by linearity.

Lemma A14: Let $f \in c^\infty(U,p)$, Then there exist functions $g_{i}, \ldots\ldots, g_n \in c^\infty(U,p)$and a neighborhood $W \subset dom(f) \cap dom(g_1) \ldots\ldots\ldots\ldots \cap dom(g_n)$ of *p* such that

(a) $f(x) = f(p) + \sum_{i=1}^{n} \left(x^i - x^i(p)\right) g_i(x)$, $\forall x \in W$;

(b) $g_i(p) = \frac{\partial f}{\partial x^i}(p), \forall 1 \leq \mathrm{i} \leq \mathrm{n}$;

Proof. Let us define

$$g_i(x) = \int_0^1 \frac{\partial f}{\partial x^i}(t(x-p)+p)dt,$$

In order to prove (2), we consider

$$g_i(p) = \int_0^1 \frac{\partial f}{\partial x^i}(p)dt = \frac{\partial f}{\partial x^i}(p)\int_0^1 dt$$

$$\therefore g_i(p) = \frac{\partial f}{\partial x^i}(p), \forall 1 \leq \mathrm{i} \leq \mathrm{n};$$

Again to prove (1), we consider

$$f(x) - f(p) = \int_0^1 \frac{d}{dt}(f(t(x-p)+p))dt$$

$$= \int_0^1 \{\sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(t(x-p)+p)(x^i - x^i(p))\}dt$$

$$= \sum_{i=1}^{n} \{\int_0^1 \frac{\partial f}{\partial x^i}(t(x-p)+p)dt\}(x^i - x^i(p))$$

$$= \sum_{i=1}^{n} g_i(x^i - x^i(p))$$

$$\therefore f(x) = f(p) + \sum_{i=1}^{n} \left(x^i - x^i(p)\right) g_i(x), \forall x \in W \;;$$

B. Smooth Maps and their Differentials

Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be open subsets. We consider functions $\phi: U \to V$ and their coordinate representations $\phi = (\phi^1, \phi^2 \dots \dots, \phi^m)$,

where each $\phi^i: U \to \mathbb{R}$.

Definition B1: If $\phi^i \in c^k(U), 1 \leq \mathrm{i} \leq \mathrm{m}$, then we say that$\phi: U \to V$ is a map of *class*$c^k (0 \leq \mathrm{k} \leq \infty)$. If $\phi$is of class c^∞ then it is called a *smooth map*.

Definition B2: If $\phi: U \to V$ is *smooth* and if $p \in U$, let

$$d\phi_p = \phi_{*p}: T_p(U) \to T_{\phi(p)}(V)$$

be defined by $\phi_{*p}\langle s\rangle_p = \langle \phi \circ s\rangle_{\phi(p)}$,

for arbitrary$\langle s\rangle_p \in T^p(U)$. This is called the*differential*[1] of ϕ at p.

Theorem B3: If $\phi: U \to V$ is smooth, if $p \in U$ and if $D_p \in T_p(U)$ and if is viewed on a derivative of $\mathcal{B}_p$, then $D_{\phi(p)}: \mathcal{B}_{\phi(p)} \to$

$\mathbb{R}$, defined$D_{\phi(p)}[f]_{\phi(p)} = D_p[f \circ \phi]_{\ p}$,is an element of $T_{\phi(p)}(V)$. We define

$$d_{\phi(p)}(D_p) = \phi_{*p}(D_p) = D_{\phi(p)}$$

Proof. At first we prove that$D_{\phi(p)}$ is linear. If $t \in \mathbb{R}$ then

$$D_{\phi(p)}(t[f]_{\phi(p)}) = D_{\phi(p)}[tf]_{\phi(p)}$$

$$= D_p[(tf) \circ \phi]_p$$

$$=D_p(t[f \circ \phi]_p)$$

$$= tD_p[f \circ \phi]_{\ p}$$

$$= tD_{\phi(p)}[f]_{\ \phi(p).}$$

$$\therefore D_{\phi(p)}(t[f]_{\phi(p)}) = tD_{\phi(p)}[f]_{\ \phi(p).}$$

Similarly,

$$D_{\phi(p)}([f]_{\phi(p)}[g]_{\phi(p)}) = D_{\phi(p)}[f]_{\phi(p)} + D_{\phi(p)}[g]_{\phi(p)}\ .$$

It remains to prove the Leibnitz rule:

$$D_{\phi(p)}([f]_{\phi(p)}[g]_{\phi(p)}) = D_{\phi(p)}[fg]_{\phi(p)}$$

$$= D_p[(fg) \circ \phi]_p$$

$$= D_p[(f \circ \phi)(g \circ \phi)]_p$$

$$= D_p([f \circ \phi]_p)g\big(\phi(p)\big) + f(\phi(p))D_p([g \circ \phi]_p)$$

so,$D_{\phi(p)}([f]_{\phi(p)}[g]_{\phi(p)}) = ([f]_{\phi(p)}g\big(\phi(p)\big) + f(\phi(p))D_{\phi(p)}([g]\phi_{(p)})$

This completes the proof.

C. Diffeomorphisms and Maps of Constant Rank

If $\Phi: U \to V$ is a diffeomorphism between open subsets of $\mathbb{R}^n$, then the Jacobean matrix $J\Phi$ (p) is non singular , for all $p \in U$. While the converse is not exactly true, it is true locally.

Definition C1: A *smooth map*$\Phi: U \to V$, between open subsets of Euclidean spaces of possibly different dimensions, has *constant rank* k if the rank of the linear map $d\Phi_x: T_x(U) \to T_{\phi(x)}(V)$ is k at every point of U. Equivalently the *Jacobian matrix*$J\Phi$ has constant rank k on U.

Example C2: Consider the following composition where $k < n, k < m$

$\mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k \to \mathbb{R}^m$,

and

$$\pi\left(x^1, \dots \dots, x^k, y^1, \dots \dots, y^{n-k}\right) = \left(x^1, \dots \dots, x^k\right)$$

$$i\left(x^1, \dots \dots, x^k\right) = (x^1, \dots \dots, x^k, 0, \dots \dots, 0)$$

The Jacobian of $i \circ \pi$ is constantly the $n \times m$ matrix having $I_k$as its upper left $k \times$ kcorner and zeros elsewhere, The rank is constantly$k$.

Theorem C3: Let $\Phi: U \to V$ be smooth, where $U, V \subset \mathbb{R}^n$ are open subsets and let $p \in U$.. If is a linear isomorphism, then there is an open neighborhood of in such that is a diffeomorphism of onto an open neighborhood in V.

Definition C4: A smooth map $\Phi: U \to V$ is said to be *submersion* if it has constant rank *n* on $U$and it is said to be an *immersion* if it has constant rank *m* on *V*, where $U \subseteq \mathbb{R}^n$and $V \subseteq \mathbb{R}^n$are open subsets.

D. Smooth Submanifold

Let $U \subseteq \mathbb{R}^n$be an open set. A topological subspace $N \subseteq U$is said to be a smooth manifold of $U$of dimension $r \leq n$if, for each $x \in N$, $\exists U_x \subseteq U$, an open neighborhood of *x*,and a diffeomorphism$f: U_x \to Q$ onto an open subset$Q \subseteq \mathbb{R}^n$ such that $f(N \cap U_x) = Q \cap \mathbb{R}^r$.

Definition D1:A topological space *X* is said to be *locally Euclidean* if, for every$x \in$ X, $\ni n \geq 0$ an open neighborhood of $U \subseteq X$ of $x,$an open subset$W \subseteq \mathbb{R}^n$ and a homeomorphism$\phi: U \to$ W .

Lemma D2: If $N \subseteq U$ is a smooth submanidfold of dimension r, then N is also a topological sub manifold of dimension $r$of U.

Proof. Here we use the notation of the above definition. If $N \subseteq U$ is a smooth submanifold of dimension r then we have to show that $N$is also a topological submanifold of dimension $r$of U. Then $N \cap U_x$ is an open neighborhood of x in the relative topology of Nin U, for all$x \in N$. Since $f$carries $N \cap U_x$homeomorphically onto the open subset $Q \cap \mathbb{R}^r$of $\mathbb{R}^r$, it follows that N, with the relative topology, is locally Euclidean of dimension r, the inclusion map $i: N \to U$being a topological imbedding. As a topological subspace of Euclidean space, N is Hausdorff second countable.

Definition D3: If $N \subseteq U$ is an $r-$*dimensional smooth submanifold* of the open set $U \subseteq \mathbb{R}^n$, and if $x \in N$ a vector $v \in T_x(U)$*is tangent* to N at x if, as an *infinitesimal curve*, $v = \langle s \rangle_x$has a representative $s: (-\varepsilon, \varepsilon) \to U$such that $s(t) \in N, -\varepsilon < t < \varepsilon$.The subset consisting of all vectors tangent to N at x, is called the *tangent space* to N at x.

Lemma D4: If $N \subseteq U$ is an r-dimensional smooth submanifold of the open set $U \subseteq \mathbb{R}^n$, and if $x \in N$, the tangent space $T_x(N)$ is an r-dimensional vector subspace of $T_x(U)$.

Proof. Let $U \subseteq \mathbb{R}^n$ be an open subset. For the 'model' case $N = \mathbb{R}^r \subseteq \mathbb{R}^n$, the assertion is evident. We use all notation given above definition. If the smooth path $s: (-\varepsilon, \varepsilon) \to U$ has image in $N$and $s(0) = x$, then the diffeomorphism$f: U_x \to Q$ sends $s$to a smooth path $f \circ s$in $\mathbb{R}^r$through $f(x)$. Thus the linear isomorphism

$$df_x: T_x(U_x) \to T_{f(x)}(Q).$$

Carrier $T_x(N)$ into the vectorspace $T_{f(x)}(Q \cap \mathbb{R}^r) = \mathbb{R}^r$. .But $Q \cap \mathbb{R}^r$is mapped onto $U_x \cap N$ by $f^{-1}$and the same argument shows that the inverse isomorphism $d(f^{-1})_{f(x)} = (df_x)^{-1}$ carries $T_{f(x)}(Q \cap \mathbb{R}^r)$ into $T_x(N)$. Thus the assertion follows.

E. Local Flows

A local flow Φ around $x_0 \in U$ is a smooth map

$$\Phi: (-\varepsilon, \varepsilon) \times W \to U,$$

where W is a suitable open neighborhoodof x_0 in U, such that

(a) $\Phi_0 : W \to U$ is the inclusion $W \to U$;

(b) $\Phi_{t_{1+t_2}}(x) = \Phi_{t_1}(\Phi_{t_2}(x))$

whenever both sides of this equation are defined. If $z \in U$, the flow line through z is the curve

$$\sigma(t) = \Phi_t(z), -\varepsilon < t < \varepsilon\,.$$

Definition E1: If Φ is a local flow on U and $q \in U$, curves of the form

$S_q^{\alpha}(t) = \Phi_t^{\alpha}(q), -\varepsilon_{\propto} < t < \varepsilon_{\propto}$,

Where $q \in V_{\propto}$,,are called flow lines[4] of Φ through q.

Lemma E2: Every vector field $X \in \chi(U)$ is the infinitesimal generator of a local flow Φ on U. If two local flows Φ and ψ have the same infinitesimal generator *X*, then Φ ∪ ψ is a local flow with the same infinitesimal generator *X*.

Definition E3: Let $s: (a, b) \to U$be smooth. The velocity vector of s at $t_0 \in (a, b)$ is

$$s(\dot{t_0}) = s_{*t_0}(\frac{d}{dt}\Big|t_0) \in T_{s(t_0)}(U)$$

Definition E4: The map $\dot{s}: (a, b) \to T(U)$is called the *velocity field* of the smooth curve

$$s: (a, b) \to U.$$

Definition E5: Let $X \in \chi(U)$and $x_0 \in U$. Then there is a local flow around x_0 such that the flow lines are integral curves to *X*. Two such local flows agree on their common domain.

Definition E6: The local flow Φ is associated to $\in \chi(U)$, where the vector field *X* is called the *infinitesimal generator* of the local flow Φ .

Theorem E7: Given $f \in C^{\infty}(U)$, there is a function $g \in C^{\infty}((-\varepsilon, \varepsilon) \times W$ such that

(a) $f(\Phi_{-t}(x)) = f(x) - tg(t, x), \forall x \in W, -\varepsilon < t < \varepsilon$;

$$(b) X_x(f) = g(0, x) \forall x \in W$$

Proof. Let us define

$h(t, x) = f(\Phi_{-t}(x)) - f(x)$

So, we have $h \in C^{\infty}((-\varepsilon,\varepsilon) \times W$ and $h(0,x) = 0$.To simplify the notation, we denote the partial of h with respect to t by $h(\dot{t},x)$.We also define $g \in C^{\infty}((-\varepsilon,\varepsilon) \times W$ by

$$g(t,x) = \int_0^1 \dot{h}\,(tu,x)\,du$$

Then

$$-tg(t,x) = \int_0^1 \dot{h}\,(tu,x)t\,du$$

$$= \int_0^1 \dot{h}\,(v,x)\,dv$$

$$= h(t,x) - h(0,x)$$

$$= f(\Phi_{-\mathrm{t}}(x)) - \mathrm{f(x)}\,,$$

$$\therefore f(\Phi_{-\mathrm{t}}(x)) = \mathrm{f(x)} - \mathrm{tg(t,x)}\,, \forall x \in W, -\varepsilon < t < \varepsilon;$$

which gives the first assertion.

Now, for the second we consider the following :

$$g(0,x) = \lim_{t \to 0} g(t,x)$$

$$= \lim_{t \to 0} \frac{f(\Phi_{-\mathrm{t}}(x)) - \mathrm{f(x)}}{-\mathrm{t}}$$

$$= X_x(f)$$

$$\therefore X_x(f) = g(0,x),)\forall x \in W$$

So, the second part is complete.

Hence the theorem is proved.

2. CONCLUSION

We have focused some important preliminaries and fundamental definitions, examples, lemmas and theorems which is essential to present this paper. By using Euclidean space,Classes Differentiability, Smooth Maps and their Differentials, Diffeomorphisms and Maps of Constant Rank, Smooth Submanifoldand topological sub manifoldthe Lemma D2 is established and so on.

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