A Comparative Study by Numerical Solutions of Volterra Integral Equations Using Legendre, Laguerre and Hermite Polynomials

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Abstract: ***In this paper, a method is presented for solving of linear Volterra integral equations (VIE) of both first and second kind including Abel's integral equation by applying a very simple and efficient Galerkin weighted residual method using Legendre, Laguerre and Hermite polynomials. Some errors of an approximate solution are compared to the exact solution by using these polynomials considering numerical examples of Volterra integral equations to this paper.***

Keywords: Volterra equation; Legendre, Laguerre and Hermite

1. Introduction

Many researchers were solved numerically of some linear and nonlinear integral equation of both first and second kinds by Mandal and Bhattacharya [11] using Bernstein polynomials. For finding approximate solutions of Volterra integral equations were also presented by Maleknejad et al [4] using Bernstein's approximation and Changqing Yang et al [1] using Laplace transform. By using Hermite and Chebyshev Polynomials, Rahman and Islam [2] were compared to approximate solutions of Volterra integral equations. Shahsavaran[5] obtained numerical solution of Volterra integral equations by Collocation Method using Block-Pulse Functions and Taylor Expansion.

Five illustrative examples of linear and nonlinear Volterra integral equations have been solving numerically by the technique of very well-known Galerkin method using Legendre, Laguerre and Hermite piecewise polynomials which are in the basis trial function. Finally I have compared to errors of approximate solutions of VIE by accuracy and efficiency.

2. The Polynomial Bases

2.1. Legendre Polynomials: The general form of the Legendre polynomials [3] of nth degree is defined by

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$$P_n(x)=\sum_{r=0}^{[n/2]}\frac{(-1)^r(2n-2r)}{2^n(r)(n-r)(n-2r)}x^{2n-r}\;;\qquad \left[\frac{n}{2}\right]=\begin{cases}n/2 & \text{if } n \text{ is even}\\ (n+1)/2 & \text{if } n \text{ is odd}\end{cases}\quad\ldots\ldots\ldots\ldots (1)$$

2.2 Laguerre Polynomials:

The general form of the Laguerre polynomials [9] of nth degree is defined by

$$L_n(x)=\sum_{r=0}^{n}\frac{(-1)^r n!}{2^n(r!)^2(n-r)!}x^r \quad\ldots\ldots\ldots (2)$$

2.3 Hermite Polynomials: The general form of the Hermite polynomials [2] of nth degree is defined by

$$H_n(x)=\sum_{r=0}^{[n/2]}\frac{(-1)^r n!}{r!(n-2r)!}(2x)^{n-2r}\quad \left[\frac{n}{2}\right]=\begin{cases}n/2 & \text{if } n \text{ is even}\\ (n-1)/2 & \text{if } n \text{ is odd}\end{cases}\quad\ldots\ldots\ldots (3)$$

3. The General Method:

In this section, first I consider the linear Volterra integral equation (VIE) of the first kind [1]-[5], given by

$$\int_a^x K(x,t)u(t)dt=f(x),\qquad a\le x\le b\quad\ldots\ldots\ldots\ldots\ldots (4)$$

where $u(x)$ is the unknown function, to be determined, $K(x,t)$ is the kernel function, continuous or discontinuous $f(x)$ being the known function satisfying $f(a)=0$.

Now I use the technique of Galerkin method, [12], to find an approximate solution $\tilde{u}(x)$ of (4). For this, I assume that

$$\tilde{u}(x)=\sum_{i=0}^{n}c_iN_i(x)\qquad\ldots\ldots\ldots\ldots (5)$$

where $N_i(x)$ are Legendre, Laguerre and Hermite polynomials of degree i defined in equation (1-3), c_i are unknown parameters, to be determined and n is the number of piecewise polynomials. An approximate solution $\tilde{u}(x)$ will not produce an identically zero function but a function called the residual function. Substituting (5) into (4), I get the residual function as,

$$R(x)=\sum_{i=0}^{n}c_i\int_a^x K(x,t)N_i(t)dt-f(x),\qquad a\le x\le b\quad\ldots\ldots\ldots (6)$$

Now the Galerkin equations of (4) corresponding to the approximation (5), given by

$$\int_a^x R(x)N_j(t)dx = 0 \text{ ....(7)}$$

Using (6) and (7) after minor simplification, I obtain

$$\sum_{i=0}^{n} c_i \int_a^b \left[\int_a^x K(x,t)N_i(t)dt \right] N_j(x)dx \quad = \int_a^x N_j(x)f(x)dx, j = 0,1,2,3,\ldots n \quad \text{.........}$$

(8)

The above equations (8) are equivalent to the matrix form

$$DC = B \qquad (9)$$

where the elements of the matrix C, D and B are and $c_i, d_{i,j}$ and b_j respectively, given by

$$c_i = [c_1, c_2, c_3, c_4, \ldots\ldots\ldots c_n]^T$$

$$d_{i,j} = \int_a^b \left[\int_a^x K(x,t)N_i(t)dt \right] N_j(x)dx,\ i, j = 0,1,2,3,\ldots\ldots n$$

$$b_j = \int_a^x N_j(x)f(x)dx, j = 0,1,2,3,\ldots.n \qquad \text{..................... (10)}$$

Now the unknown parameters c_i are determined by solving the system of equations (10) and substituting these values of parameters in (5), I get the approximate solution $\tilde{u}(x)$ of the integral equation (4).

Now I consider the linear Volterra integral equation (VIE) of the second kind [1] – [5] given by

$$u(x) + \lambda \int_a^x K(x,t)u(t)dt = f(x), \quad a \le x \le b \quad \text{............. (11)}$$

where $u(x)$ is the unknown function to be determined, $K(x,t)$ is the kernel function, continuous or discontinuous, $f(x)$ being the known function and λ is the constant. Then applying the same procedure as described above, I obtain the matrix form

$$DC = B \quad \text{............ (12)}$$

where the elements of the matrix C, D and B are and $c_i, d_{i,j}$ and b_j respectively, given by

$$c_i = [c_1, c_2, c_3, c_4, \ldots\ldots\ldots c_n]^T$$

$$d_{i,j} = \int_a^b \left[\int_a^x K(x,t) N_i(t) dt \right] N_j(x) dx,\ i, j = 0,1,2,3,......n$$

$$b_j = \int_a^x N_j(x) f(x) dx,\ j = 0,1,2,3,....n \qquad .... (13)$$

Now the unknown parameters c_i are determined by solving the system of equations (13) and substituting these values of parameters in (5), I get the approximate solution $\tilde{u}(x)$ of the integral equation (11).

The absolute error for this formulation is defined by, Absolute Error$= |u(x) - \tilde{u}(x)|$. The formulation for linear integral equation will be discussed by considering numerical problems in the next section.

4. Illustrative Examples

Here, I am giving some examples of VIE for finding the approximate solution using above mentioned methods, which include one first kind and four second kind linear Volterra integral equations with three regular kernels, one convolution kernel and one weakly singular kernel.

4.1 Example 1: Consider the Volterra integral equation with a convolution kernel of the first kind [8]

$$\int_0^x Cos(x-t) u(t) dt = xSin(x),\ 0 \le x \le 1 \ .... (14)$$

the exact solution is $u(x) = 2Sin(x)$

Results have been shown in Table 1for n=10 using the aforementioned three polynomial bases. The absolute errors between the exact solution and the approximate solution are graphed for different values of x in Figures 1. This graph has shown that the error of Legender polynomials is up other two polynomials and these three lines are continuous from 0 to 1. The Legendre, Laguerre and Hermite polynomials basis are shown the different absolute errors have been taken statistically in the order 10^{-7} , 10^{-7} and 10^{-8} respectively which is very approach to zero.

4.2 Example 2: Consider the second kind integral equation of the form [10]

$$u(x) + \int_0^x (x-t) u(t) dt = x - x^2 + \frac{x^3}{6} - \frac{x^4}{12},\ 0 \le x \le 1 .... (15)$$

the exact solution is $u(x) = x - x^2$ Using Legendre, Laguerre and Hermite polynomials of the equation (16) for n=10, I get the approximate solution is $u(x) = x - x^2$, which is the exact solution. On the other hand, the absolute errors were obtained in the order of 10^{-7} for n=10 using Sinc approximation by J. Rashidinia & M. Zarebnia[10]

4.3 Example 3: Consider the Volterra integral equation of the second kind of the form [10]

$$u(x)+\int_0^x (x-t)u(t)dt=1-x+\frac{x^2}{2},\ 0\le x\le\frac{\pi}{2} \ldots\ldots\ldots (16)$$

the exact solution is $u(x)=1-Sin(x)$

Results have been shown in Table: 2 for n=10 using the aforementioned three polynomial bases. The absolute errors between the exact solution and the approximate solution are graphed for different values of x in Figures 2. This graph has shown that the error of Legender polynomials is up other two polynomials and these three lines are continuous from 0 to 1. The Legendre, Laguerre and Hermite polynomials basis are shown the different absolute errors have been taken statistically in the order 10^{-9}, 10^{-7} and 10^{-10} respectively which is very close to zero. On the other hand, the absolute errors were obtained in the order of 10^{-4} for n=10 using Sinc approximation by J. Rashidinia & M. Zarebnia[10]

4.4 Example 4: Consider the second kind Abel's integral equation [3]

$$u(x)-\int_0^x \frac{1}{\sqrt{x-t}}u(t)dt=x^7(1-\frac{4096}{6435}\sqrt{x}),\ 0\le x\le 1 \ldots\ldots\ldots\ldots (17)$$

the exact solution is, $u(x)=x^7$.

Using Legendre, Laguerre and Hermite polynomials of the equation for n=10, I get the approximate solution is $u(x)=x^7$., which is the exact solution. On the other hand, the absolute errors were obtained in the order of 10^{-7} for n=10 using Bernstein polynomials by Mandal B. N. and Bhattacharya S [11], in the order of 10^{-16} using Legendre polynomial by Rahman M. A. and Islam M. S. [3] which is very close to the exact result.

4.5 Example 5: Consider the Volterra integral with a convolution kernel given by [2]

$$u(x)+\int_0^x Cos(x-t)u(t)dt=Sin(x),\ 0\le x\le 1 \ldots\ldots\ldots\ldots (18)$$

the exact solution is, $u(x)=\frac{2}{\sqrt{3}}Sin(\sqrt{3}x/2).e^{-\frac{x}{2}}$

Results have been shown in Table 3 for n=10 using the aforementioned three polynomial bases. The absolute errors between the exact solution and the approximate solution are graphed for different values of x in Figures 3. This graph has shown that the error of Laguerre polynomials is up other two polynomials which have shown on axis line and the line of the error of Laguerre polynomials is continuous from 0 to 1. The Legendre, Laguerre and Hermite polynomials basis are shown the different absolute errors have been taken statistically in the order 10^{-7} , 10^{-10} and 10^{-8} respectively which is very close to zero. On the other hand, the absolute errors were obtained in the order of 10^{-8} using Laplace transformation by Changqing Yang & Jianhua Hou [1]

5. Conclusion

The numerical results shown that the presented method has approached of all four linear VIE with first kind, second kind; regular kernel, weakly singular kernel and convolution kernel by applying very simple and efficient Galerkin weighted residual method using the three polynomials as trial basis for finding the approximate solution harmony with the exact solutions. The comparative study is that as per performance of accomplishing the numerical solution of some linear VIE to the closeness of the exact solution of three polynomials, Laguerre polynomial is the best polynomial of other two Hermite and Legendre polynomials; again Hermite polynomial is better than Legendre polynomial. Finally I think that for finding the approximate solution of VIE by applying Galerkin weighted residual method using some polynomials as trial basis will be widely used in mathematical application.

References

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Appendices

Table-1

Value x	Exact Value	Error Legender	Error Laguerre	Error Hermite
0.0	0	1.820478442670497E-6	3.707971882871419E-7	2.30915326346337E-7
0.1	0.199666833293656	3.019155750683122E-7	1.184870222492584E-7	5.68081501284289E-8
0.2	0.397338661590122	5.982795798775697E-7	1.197701878785650E-7	8.97074676209364E-8
0.3	0.591040413322679	2.810269545960864E-7	5.094439381458926E-8	2.13306487251152E-7
0.4	0.778836684617301	5.03241040283875E-7	1.207010076331727E-7	5.28904644525596E-8
0.5	0.958851077208406	3.741724632355669E-7	8.672377682650279E-8	2.97723464659505E-7
0.6	1.129284946790071	5.528613162830709E-7	1.119310095720749E-7	9.67801252471645E-8
0.7	1.288435374475382	5.707035515278847E-7	1.506793885219792E-7	4.54593754861676E-7
0.8	1.434712181799045	5.015386694839918E-7	1.522681203347531E-7	2.98069535720202E-8
0.9	1.566653819254966	1.618947422787897E-6	2.022648566413210E-7	9.47338981482914E-7
1.0	1.682941969615793	0.00001134318449813	2.066155024849081E-6	7.22323103952149E-6

Table-2

Value x	Exact Value	Error Legender	Error Laguerre	Error Hermite
0.0	1	6.656405981964042E-9	9.986713296328276E-7	2.128596410244654E-9
0.1	0.90016658335317	5.61938995335964E-10	4.119275743286188E-7	4.733987091221081E-10
0.2	0.80133066920493	1.724668852176591E-9	4.248729146283381E-10	6.004434727202579E-10
0.3	0.70447979333866	1.142198557957385E-9	3.184520112053945E-7	3.382527591355711E-11
0.4	0.61058165769134	1.187624276699494E-9	1.908884878498717E-7	4.724697855174043E-10
0.5	0.52057446139579	9.23602527791445E-10	1.195197206760312E-7	1.67937330708412E-10
0.6	0.43535752660496	1.301680541132555E-9	2.925463093994551E-7	3.160893768949790E-10
0.7	0.35578231276230	3.71001118715241E-10	2.008245226781113E-7	3.246320989802598E-10
0.8	0.28264390910047	1.385251802155096E-9	5.88503009657515E-8	7.360545506429617E-11
0.9	0.21667309037251	2.55101384460942E-10	2.690900363644388E-7	3.389355462957155E-10
1.0	0.15852901519210	1.260346271791945E-9	2.500006467398563E-7	1.463715815219757E-10

Table-3

Value x	Exact Value	Error Legender	Error Laguerre	Error Hermite
0.0	0	5.098031396755242E-7	1.991415232716065E-8	1.5016185703209395E-7
0.1	0.09500408335292	1.189055968409125E-7	4.861272370759728E-9	3.612199178659736E-8
0.2	0.18006400247621	1.663567890664019E-7	6.048788997459198E-9	4.7736134295428556E-8
0.3	0.25531729176720	1.319277714850741E-8	1.29881327914915E-10	1.6139981018348237E-9
0.4	0.32098164218657	1.520163611656855E-7	5.191320429087653E-9	4.2913501030472645E-8
0.5	0.37734520347490	5.880330666840905E-9	5.06176933701141E-10	5.174047301714779E-10
0.6	0.42475713928988	1.543624366662221E-7	4.691275479196122E-9	4.22560730850563E-8
0.7	0.46361850100044	2.112981845048267E-9	8.07714395367753E-10	2.5474169995653995E-9
0.8	0.49437347658866	1.705632147741553E-7	4.689407639979493E-9	4.58746624576456E-8
0.9	0.51750106217150	1.313902323607507E-7	2.744402149978953E-9	3.353248301252165E-8
1.0	0.53350719511469	5.356258405697645E-7	1.234443192110745E-8	1.4013144200575312E-7

Figure-1

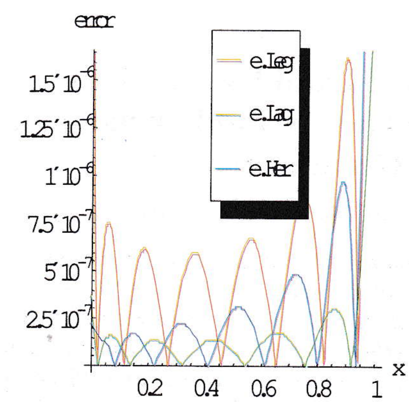

Figure-2

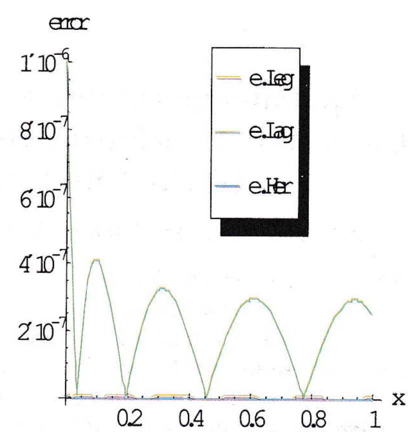

Figure-3