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Computing Determinants of Block Matrices

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ABSTRACT*: *In some studies of physics and applied mathematics, there arise large size of matrices and calculating determinants of those matrices are very complex. In this case we can partition on such matrices into some blocks. After partitioning, the new matrix which elements are those partitions is a block matrix. In this article, we have studied and explored some formulae to compute the determinant of block matrices. We have curbed our absorption in 2×2 block matrices, where each blocks are any $m \times n$ size, where $m, n \in \mathrm{N}$.

Keywords: Block matrix, Block diagonal matrix, Schur complement, Determinant.

1. INTRODUCTION

Block matrices appear frequently in physics and applied mathematics [1-5]. Among those some of the determinants of these matrices are very large, for example, a model of high density quark matter must include color (3), flavor (2-6), and Dirac (4) indices, giving rise to a matrix between size 24× 24 and 72 ×72. In this case, calculating determinants of those matrices are very complex such as computational time and technique. But, we can calculate the determinant easily if we partition these matrices into some blocks. Silvester [6] has calculated the determinant of $m \times m$ block matrices. Block matrices also have been studied by Molinari and Popescu [9, 10]. In this work we have studied and investigated some properties of 2×2 block matrices. These properties of 2×2 block matrices can help to calculate the determinant of any large sizes matrices.

The paper is organized as follows. In section 2 we have discussed about basic definitions and notations which are used throughout this paper. In section 3, we have studied and investigated some formulae to compute the determinant of 2×2 block matrices with an example.

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2. PRELIMINARIES

Definition 2.1: [7] A block matrix (also called partitioned matrix) is a matrix of the kind

$$A = \begin{bmatrix} B & C \\ D & E \end{bmatrix}$$

Where B, C, D and E are also matrices, called blocks. Basically, a block matrix is obtained by cutting a matrix two times: one vertically and one horizontally. Each of the four resulting pieces is a block.

Example 2.1 (a): We consider the matrix

$$A = \begin{bmatrix} 3 & 1 & 3 \\ 2 & 5 & 7 \\ 1 & 2 & 3 \end{bmatrix}$$

We can partition it into four blocks as

$$A = \begin{bmatrix} 3 & 1 & \vdots & 3 \\ 2 & 5 & \vdots & 7 \\ \cdots & \cdots & \vdots & \cdots \\ 1 & 2 & \vdots & 3 \end{bmatrix}$$

By taking

$$B = \begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 3 \\ 7 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 3 \end{bmatrix}, \quad E = \begin{bmatrix} 3 \end{bmatrix}$$

The above matrix can be written as

$$A = \begin{bmatrix} B & C \\ D & E \end{bmatrix}$$

Definition 2.2: [7] Block matrices whose off-diagonal blocks are all equal to zero are called block-diagonal. The matrix $A = \begin{bmatrix} B & 0 \\ 0 & E \end{bmatrix}$ is a block diagonal where 0 is a zero matrix.

Definition 2.3: [7] Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where a, b, c, d are numbers, then the determinant of A is $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$

3. DETERMINANTS OF BLOCK MATRICES

Proposition 3.1: Let $A=\begin{bmatrix} B & 0 \\ 0 & E \end{bmatrix}$ be a block diagonal matrix, where B and E are of any $n\times n$ and $m\times m$ size where $m\neq n; m,n\in N$ and 0 is zero matrix, then $|A|=\begin{vmatrix} B & 0 \\ 0 & E \end{vmatrix}=|B||E|$.

Proof: Let I be a $n\times n$ matrix. Then

$$\begin{bmatrix} B & 0 \\ 0 & E \end{bmatrix}=\begin{bmatrix} B & 0 \\ 0 & I_m \end{bmatrix}\begin{bmatrix} I_n & 0 \\ 0 & E \end{bmatrix}$$

Now by the product formula, we have

$$\begin{vmatrix} B & 0 \\ 0 & E \end{vmatrix}=\begin{vmatrix} B & 0 \\ 0 & I_m \end{vmatrix}\times\begin{vmatrix} I_n & 0 \\ 0 & E \end{vmatrix}$$

$$=|B||E|$$

Proposition 3.2: Let $A=\begin{bmatrix} B & C \\ D & E \end{bmatrix}$ be a block matrix, if $C=0$, or $D=0$ that is $A=\begin{bmatrix} B & 0 \\ D & E \end{bmatrix}$ or $A=\begin{bmatrix} B & C \\ 0 & E \end{bmatrix}$, then $\begin{vmatrix} B & 0 \\ D & E \end{vmatrix}=|B||E|=\begin{vmatrix} B & C \\ 0 & E \end{vmatrix}$

Proof: Trivial

Proposition 3.3: Let $A=\begin{bmatrix} B & C \\ D & E \end{bmatrix}$ be a block matrix, if $A=0$ or $E=0$ and C, or D is square but not of same size, then

$$\begin{vmatrix} 0 & C \\ D & E \end{vmatrix}=|D||C|=\begin{vmatrix} B & C \\ D & 0 \end{vmatrix}$$

Proof: Trivial

Proposition 3.4: Let $A=\begin{bmatrix} B & C \\ D & E \end{bmatrix}$ be a block matrix, if $A=0$ or $E=0$ and C, or D is square and of same size, then

$$\begin{vmatrix} 0 & C \\ D & E \end{vmatrix}=|-D||C|=\begin{vmatrix} B & C \\ D & 0 \end{vmatrix}$$

Proof: Trivial

Definition 3.5: [8] If $A=\begin{bmatrix} B & C \\ D & E \end{bmatrix}$, then $S_B = E - DB^{-1}C,$ $S_C = D - EC^{-1}B,$ $S_D = C - BE^{-1}F,$ $S_E = B - CE^{-1}D,$ are called the Schur complement of B, C, D and E respectively.

Theorem 3.6: Let $A=\begin{bmatrix} B & C \\ D & E \end{bmatrix}$ be a block matrix, where the block, B, C, D and E are of any $m \times n$ size where $m, n \in N.$ If B is non-singular then $|A| = |B||S_B|$, where S_B is the Schur complement of B and also if B and E are of same size then $|A| = |BE| - |DC|$, if $BD = DB$; and $|A| = |EB| - |DC|$, if $BC = CB$.

Proof: Suppose $A=\begin{bmatrix} B & C \\ D & E \end{bmatrix}$ and B is non-singular.

Therefore, $\begin{bmatrix} I & 0 \\ -DB^{-1} & I \end{bmatrix}\begin{bmatrix} B & C \\ D & E \end{bmatrix} = \begin{bmatrix} B & C \\ 0 & E - DB^{-1}C \end{bmatrix}$

$$\Rightarrow |I||A| = |B||E - DB^{-1}C|$$

$$\text{Thus } |A| = |B||E - DB^{-1}C| = |B||S_B| \qquad (3.1)$$

$$= |BE - BDB^{-1}C|$$

If B and E are of same size.

$$= |BE - DC| \text{ if } BD = DB \qquad (3.2)$$

Again we can write (3.1) as $|A| = |E - DB^{-1}C||B|$

$$= |EB - DB^{-1}CB|$$

If B and E are of same size.

$$= |BE - DC| \text{ if } BC = CB \qquad (3.3)$$

Theorem 3.7: Let $A=\begin{bmatrix} B & C \\ D & E \end{bmatrix}$ be a block matrix, where the block, B, C, D and E are of any $m \times n$ size where $m, n \in N.$

If C is non-singular and also if C and D are not of same size then $|A| = |C||S_C|$, where S_C is the Schur complement of C, again if C and D are of same size then $|A| = -|C||S_C|$ and $|A| = |EB - CD|$ if $CE = EC$; also $|A| = |EB - DC|$ if $BC = CB$.

Proof: Suppose $A = \begin{bmatrix} B & C \\ D & E \end{bmatrix}$ and C is non-singular.

Therefore, $\begin{bmatrix} I & 0 \\ -EC^{-1} & I \end{bmatrix}\begin{bmatrix} B & C \\ D & E \end{bmatrix} = \begin{bmatrix} B & C \\ D - EC^{-1}B & 0 \end{bmatrix}$ (3.4)

If C and D are not of same size then we obtain

$$\Rightarrow |I||A| = |D - EC^{-1}B||C|$$

$$\text{Thus } |A| = |C||D - EC^{-1}B| = |C||S_C| \tag{3.5}$$

Now from (3.4) if C and D are of same size then we obtain

$$\Rightarrow |I||A| = -|D - EC^{-1}B||C|$$

$$\text{Thus } |A| = -|C||D - EC^{-1}B| = -|C||S_C| \tag{3.6}$$

$$= |CEC^{-1}B - CD|$$

$$= |EB - CD| \text{ if } CE = EC \tag{3.7}$$

Again we can write (3.5) as $|A| = -|D - EC^{-1}B||C|$

$$= |EC^{-1}BC - DC|$$

$$= |EB - DC| \text{ if } BC = CB \tag{3.8}$$

Theorem 3.8: Let $A = \begin{bmatrix} B & C \\ D & E \end{bmatrix}$ be a block matrix, where the block, B, C, D and E are of any $m \times n$ size where $m, n \in N$. If B, C, D and E are of all square matrices, then $|A| = |BE - DC|$ if ; $BD = DB$; $|A| = |EB - DC|$ if $BC = CB$; $|A| = |EB - CD|$ if $CE = EC$; $|A| = |BE - CD|$ if $DE = ED$.

Proof: Trivial

Example 3.9: Let $A = \begin{bmatrix} 1 & 1 & 0 & 4 & 5 & 6 \\ 3 & 4 & 5 & 9 & 4 & 6 \\ 3 & 2 & 1 & 4 & 3 & 1 \\ 0 & 0 & 0 & -1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 & 2 \end{bmatrix}$ be a 6×6 order matrix.

Now for computing determinant of A we can partition this matrix as follows.

Let $B = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 4 & 5 \\ 3 & 2 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 4 & 5 & 6 \\ 9 & 4 & 6 \\ 4 & 3 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $E = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 2 & 1 \\ 0 & 5 & 2 \end{bmatrix}$.

Therefore, $|A| = \begin{vmatrix} B & C \\ 0 & E \end{vmatrix} = |B||E| = 6 \cdot 1 = 6$

4. CONCLUSIONS

A block matrix or a partitioned matrix is a partition of a matrix into rectangular smaller matrices called blocks. Block matrices emerge often in modern applications of linear algebra and it is not very difficult to compute the determinant of those block matrices. But it is important, how to implement the algorithms to compute the determinant of the block matrices. In this article, we tough on a few thoughts and apparatus for computing the determinant of block matrix. Particularly, we have proved several propositions for computing determinant of 2×2 block matrices. Also by using the Schur complement, some theorems for computing determinant of 2×2 block matrices have been proved. This work will be helpful to readers as well as researchers to calculate determinant of any large sizes of matrices.

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