Planar Drawing of Bipartite Graph by Eliminating Minimum Number of Edges

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Abstract: *A graph is said to be bipartite if its vertices can be partitioned into two subsets such that each edge of graph connects a vertex of one set to another. A planar drawing is more understandable than its clumsy drawing. Not all complete bipartite graphs have planar drawings, objectives of this research is to obtain the minimum number(s) of edges that must be eliminated for finding planar drawing. A drawing algorithm for finding the planar drawings of complete bipartite graphs has been proposed and furthermore the maximum number of edges in the planar drawing of complete bipartite graph has been found.*

Keywords: Bipartite Graph, Planar Drawings, Planar Graph

I. Introduction

A graph *G* is connected if for every pair {*u*, *v*} of distinct vertices there is a path between *u* and *v*. A (connected) component of a graph is a maximal connected subgraph. A graph which is not connected is called a disconnected graph[1].

A planar drawing is a drawing of a graph in which any two edges do not intersect at any point except at their common end vertex [2]. If a graph has a planar drawing, then it is preferable to find it, because planar drawings are relatively easy to understand in comparison with the non-planar drawings. Unfortunately not every graph has a planar drawing. If a graph has a planar drawing, then it is called a planar graph.

Straight line drawing is one of the earliest graph drawing conventions. It is natural to draw each edge of a graph as a straight line between its end vertices and a drawing of a graph in which each edge is drawn as a straight line segment is called a straight line drawing [2].

In this research the properties of bipartite graph and complete bipartite graph have been studied. Mathematical proof has been derived on planar drawing and an algorithm has been developed that shows the number of edges that must be eliminated to obtain the planar drawings of bipartite graphs.

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II. Bipartite Graphs

A graph *G* is said to be bipartite if its vertices *V* can be partitioned into two subsets *M* and *N* such that each edge of *G* connects a vertex of *M* to a vertex of *N* [3]. By a complete bipartite graph, we mean that each vertex of *M* is connected to each vertex of *N*; this graph is denoted by *K*m, n where m is the number of vertices in *M* and where n is the number of vertices in *N*, and for standardization, we will assume m<=n.

Bipartite graphs are perhaps the most basic of objects in graph theory, both from a theoretical and practical point of view. However, sometimes they have been considered only as a special class in some wider contexts. Bipartite graph is illustrated with many applications especially to problems in timetabling, chemistry, communication networks and computer science.

Bipartite graphs are useful for modeling matching problems. An example of bipartite graph is a job matching problem. Suppose P is a set of people and J is a set of jobs, with not all people suitable for all jobs. This can be the model as a bipartite graph (P, J, E). If a person *px* is suitable for a certain job *jy* there is an edge between *px* and *jy* in the graph. The marriage theorem provides a characterization of bipartite graphs which allow perfect matching.

Bipartite graphs are extensively used in modern coding theory, especially to decode codeword received from the channel. Factor graphs and Tanner graphs are examples of this.

In computer science, a Petri nets a mathematical modeling tool used in analysis and simulations of concurrent systems. A system is modeled as a bipartite directed graph with two sets of nodes: A set of "place" nodes that contain resources, and a set of "event" nodes which generate and/or consume resources. Petri nets utilize the properties of bipartite directed graphs and other properties to allow mathematical proofs of the behavior of systems while also allowing easy implementation of simulations of the system.

Previous study was planar straight-line drawings of bipartite planar graphs. The results generalize to triangle-free planar graphs which admit drawings in an n/2 x (n/2-1) –grid. The improvement is to allow any kind of triangle for the planar drawings any complete bipartite graph.

A. Bipartite Graph Properties

A graph is bipartite if and only if it does not contain an odd cycle. Therefore, a bipartite graph cannot contain a clique of size 3 or more. Properties are as follows:

1. A graph is bipartite if and only if it is 2-colorable, (i.e. its chromatic number is less than or equal to 2).
2. The size of minimum vertex cover is equal to the size of the maximum matching.
3. The size of the maximum independent set plus the size of the maximum matching is equal to the number of vertices.

4. For a connected bipartite graph the size of the minimum edge cover is equal to the size of the maximum independent set.
5. For a connected bipartite graph the size of the minimum edge cover plus the size of the minimum vertex cover is equal to the number of vertices.
6. The spectrum of a graph is symmetric if and only if it's a bipartite graph.
7. Every bipartite graph is a perfect graph.

B. Complete Bipartite Graph

A complete bipartite graph *G: = (V1 + V2, E)* is a bipartite graph such that for any two vertices and *v1 v2* is an edge in *G*. The complete bipartite graph with partitions of size and is denoted $K_{m,n}$[5].

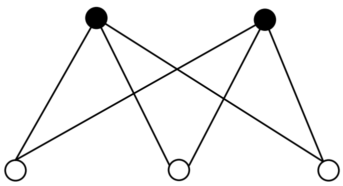

Fig 1. Complete bipartite graph *m=3, n =2*

In Fig 1 total number of Vertices are *n+m*, total edges are *mn* and automorphisms are *2m!n!* if *m=n*, otherwise *m!n!*.

C. Properties of Complete Bipartite Graph

1. Given a bipartite graph, finding its complete bipartite subgraph $K_{m,n}$ with maximal number of edges is an NP-complete problem.
2. A planar graph cannot contain $K_{3,3}$ as a minor ; an outerplanar graph cannot contain $K_{3,2}$ as a minor (These are not sufficient conditions for planarity and outerplanarity, but necessary).
3. A complete bipartite graph. $K_{n,n}$ is a Moore graph and a (*n*,4)-cage
4. A complete bipartite graph $K_{n,n}$ or *Kn*,*n* + 1 is a TurᏱn graph.
5. A complete bipartite graph $K_{m,n}$ has a vertex covering number of min{*m*,*n*} and an edge covering number of max{*m*,*n*}
6. A complete bipartite graph $K_{m,n}$ has a maximum independent set of size max{*m*,*n*}

7. The adjacency matrix of a complete bipartite graph $K_{m,n}$ has eigenvalues , and 0; with multiplicity 1, 1 and n+m-2 respectively.
8. The laplacian matrix of a complete bipartite graph $K_{m,n}$ has eigenvalues $n+m$, n, m, and 0; with multiplicity 1, $m-1$, $n-1$ and 1 respectively.
9. A complete bipartite graph $K_{m,n}$ has $mn-1$ $nm-1$ spanning trees.
10. A complete bipartite graph $K_{m,n}$ has a maximum matching of size $\min\{m,n\}$
11. A complete bipartite graph $K_{n,n}$ has a proper n-edge-coloring corresponding to a Latin square.
12. The last two results are corollaries of the Marriage Theorem as applied to a K-regular bipartite graph.

III. Planer Drawings

In graph theory, a planar graph is a graph which can be embedded in the plane, i.e., it can be drawn on the plane in such a way that its edges intersect only at their endpoints. A planar graph already drawn in the plane without edge intersections is called a plane graph or planar embedding of the graph. A plane graph can be defined as a planar graph with a mapping from every node to a point in 2D space, and from every edge to a plane curve, such that the extreme points of each curve are the points mapped from its end nodes, and all curves are disjoint except on their extreme points. Plane graphs can be encoded by combinatorial maps [6].

It is easily seen that a graph that can be drawn on the plane can be drawn on the sphere as well, and vice versa. The equivalence class of topologically equivalent drawings on the sphere is called a planar map. Although a plane graph has an external or unbounded face, none of the faces of a planar map have a particular status.

Generalization of planar graphs are graphs which can be drawn on a surface of a given genus. In this terminology, planar graphs have graph genus 0, since the plane (and the sphere) are surfaces of genus 0. See "graph embedding" for other related topics.

A. Planarity Criteria

In practice, it is difficult to use Kuratowski's criterion to quickly decide whether a given graph is planar. However, there exist fast algorithms for this problem: for a graph with n vertices, it is possible to determine in time $O(n)$ (linear time) whether the graph may be planar or not (see planarity testing).

For a simple, connected, planar graph with v vertices and e edges, the following simple planarity criteria hold:

Theorem 1: If $v \geq 3$ then $e \leq 3v - 6$;

Theorem 2: If $v > 3$ and there are no cycles of length 3, then $e \leq 2v - 4$.

In this sense, planar graphs are sparse graphs, in that they have only $O(v)$ edges, asymptotically smaller than the maximum $O(v2)$. The graph $K3,3$, for example, has 6 vertices, 9 edges, and no cycles of length 3.

Therefore, by Theorem 2, it cannot be planar. Note that these theorems provide necessary conditions for planarity that are not sufficient conditions, and therefore can only be used to prove a graph is not planar, not that it is planar. If both theorem 1 and 2 fail, other methods may be used.

IV. Results of Planer Drawings

Lemma 1: A Bipartite Graph $K_{m,n}$ has a planar drawing if $k_{m,n}$ has at most 2(m+n-2) edges.

Base Case: K_{1},n or $K_{n,1}$ $(n € N)$ is always planar.

Proof:

Let the total numbers of edges in the planar drawing of $K_{m,n}$ is denoted as $\Phi(K_{m,n})$.

The planar drawings of $K_{2,y}$ or $K_{x,2}$ without eliminating any edge has been reformed as the drawing algorithm. The complete bipartite graph of $K_{2,y}$ has its planar drawing as follow:

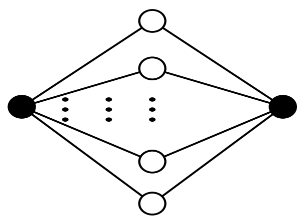

Fig 2. Planar drawing of $K_{2,y}$

So, $\Phi(K_{m,n}) = \Phi(K_{2,y})$

$= 2y$

$= 2(m+n-2)$

$K_{3,y}$ or $K_{x,3}$ is nonplanar. The aim is to find out maximum planar drawing by eliminating minimum no of edges. The complete bipartite graph of $K_{3,y}$ has its planar drawing as follow:

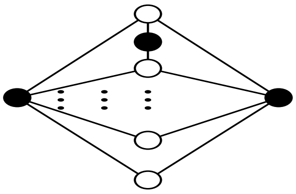

Fig 3. Planar drawing of $K_{3,y}$

So, $\Phi(K_{m,n}) = \Phi(K_{3,y})$
$= 2y+2$
$= 2y+(3–2)2$
$= 2(m+n–2)$

For a complete bipartite graph $K_{x,y}$ the planar drawing is as follow:

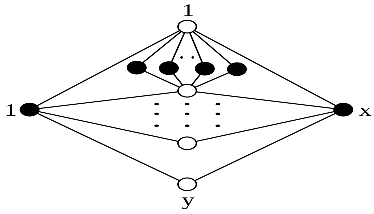

Fig 4. Planar drawing of $K_{x,y}$

So, $\Phi(K_{m,n})$
$= \Phi(K_{x,y})$
$= 2y+(x–2)2$
$= 2(x+y–2)$
$= 2(m+n–2)$

Let a complete bipartite graph is $K_{x+1,y+1}$ then its planar drawing as follow:

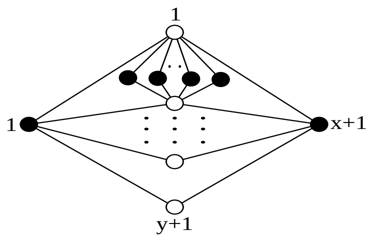

Fig 5. Planar drawing of $K_{x+1,y+1}$

So, $\Phi(K\ m,n)$
$= \Phi(K\ x,y)$
$= 2(y+1)+(x+1–2)2$
$= 2y+2+2x–2$
$=2(x+y)$
$=2(m–1+n–1)$, here $m=x+1$, $n=y+1$
$= 2(m+n–2)$

So, the planar drawing of bipartite graph has at most $2(m + n – 2)$ edges. It satisfies for every $K_{2,y}$ or $K_{x,2}$. It is also true when $K_{3,y}$ or $K_{x,3}$. For any bipartite graph $K_{x,y}$ the result is satisfied. It is found true when $x=m+1$ and $y=n+1$. So, according to the inductive method, a complete Bipartite Graph $K_{m,n}$ has a planar drawing if $K_{m,n}$ has at most $2(m+n–2)$ edges. Deducting the mentioned number of edges from total number of edges of a complete bipartite graph can obtain the minimum number of edges that must be eliminated. For finding the planar drawing of any complete bipartite graph $K_{m,n}$ at least $(m–2)(n–2)$ edges must be eliminated.

V. Theorem

Proof:

Let Ω be the total number of edges of any complete bipartite graph $K_{m,n}$. We have $K_{m,n}$ has mn edges. So $\Omega(K_{m,n})$ = mn

From lemma 1, we have the total number of edges of the planar drawing of bipartite graph $K_{m,n}$ is $\Phi(K_{m,n}) = 2(m + n - 2)$.

Let $\Delta(K_{m,n})$ be the number of edges required to obtain the planar drawing of complete bipartite graph.

$$\begin{aligned} \text{Now,}\quad \Delta(K_{m,n}) &= \Omega(K_{m,n}) - \Phi(K_{m,n}) \\ &= mn - 2(m + n - 2) \\ &= mn - 2m - 2n - 4 \\ &= m(n - 2) - 2(n - 2) \\ \Delta(K_{m,n}) &= (m - 2)(n - 2) \qquad \text{.... (proved)} \end{aligned}$$

A. Observation 1

A complete bipartite graph $K_{m,n}$ is complement of $K_{n,m}$. So, both have the same planar drawings. Thus $\Phi(K_{m,n}) = \Phi(K_{n,m})$, here Φ=total number of edges in planar drawing.

For two complete bipartite graphs $K_{m,n}$ and $K_{n,m}$, the planar drawing as follow:

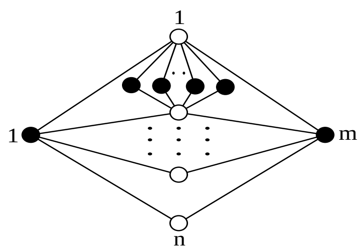

Fig 6. Planar drawing of $K_{m,n}$

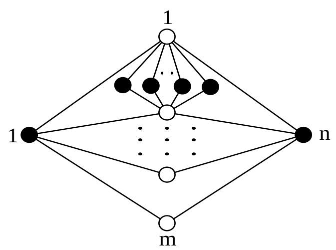

Fig 7. Planar drawing of $K_{n,m}$

Fig 6 and Fig 7 are complement. So $K_{m,n}$ and $K_{n,m}$ have the same planar drawings with same number of edges.

For m=1 and n=1, $K_{m,n}$ has already an planar drawing. If (m<3) or (n<3) or (m<3 and n<3) $K_{m,n}$ has already an planar drawing. The planar drawing of a complete bipartite graph $K_{m,n}$ will be provided where m>=3 and n>=3.

B. Observation 2

From Lemma 1 we have

$\Phi(K_{m,n}) = \Phi(K_{x,y})$
$= 2(x+y-2)$

$\Phi(K_{m,n}) = \Phi(K_{x+1,y})$
$= 2(x+1+y-2)$
$= 2(x+y-1)$
$= 2(x+y-2)+2$
$= \Phi(K_{x,y})+2$

$\Phi(K_{m,n}) = \Phi(K_{x,y+1})$
$= 2(x+y+1-2)$
$= 2(x+y-1)$
$= 2(x+y-2)+2$
$= \Phi(K_{x,y})+2$

Thus we can say $\Phi(K_{m+1,n}) = \Phi(K_{m,n+1}) = \Phi(K_{m,n}) + 2$

C. Observation 3

Let $x1+y1= x2+y2=\ldots\ldots\ldots = xn+yn = c$

$\Phi(K_{m,n}) = \Phi(K_{x1,y1})$
$= \Phi(K_{x,y})$
$= 2(x+y-2)$

$\Phi(K_{m,n}) = \Phi(K_{x2,y2})$
$= \Phi(K_{x+1,y-1})$
$= 2(x+y-2)$

$\Phi(K_{m,n})$
$= \Phi(K_{xn,yn})$
$= \Phi(K_{x+n,y-n})$
$= 2(x+y-2)$

Then $\Phi(K_{x1,y1}) = \Phi(K_{x2,y2}) = \ldots\ldots = \Phi(K_{xn,yn})$

VI. Algorithm and Flow Chart

A. Algorithm

From the above proof Algorithm for Planar drawing of Bipartite graph is stated as:

begin

step 1: Input $k_{m,n}$

step 2: if m=2 or n=2, go to step 9;

step 3: if n<m, swap(n,m);

step 4: for each i, 1<=i<=n draw all i node(s) vertically;

step 5: for each j, 1<=j<=2 draw node j left and right side of nodes drawn in step 4;

step 6: for each j, 1<=j<=2 and for each i, 1<=i<=n draw line(i,j);

step 7: for each k, 1<=k<=m-2 draw node k horizontally between i=1 and i=2;

step 8: draw line (i,k) for each i; i<=i<=2 with k; 1<=k<=m-2;
step 9: planar drawing;

end

B. Flow Chart

From the steps mentioned above the following flow chart can be provided in Fig 7.

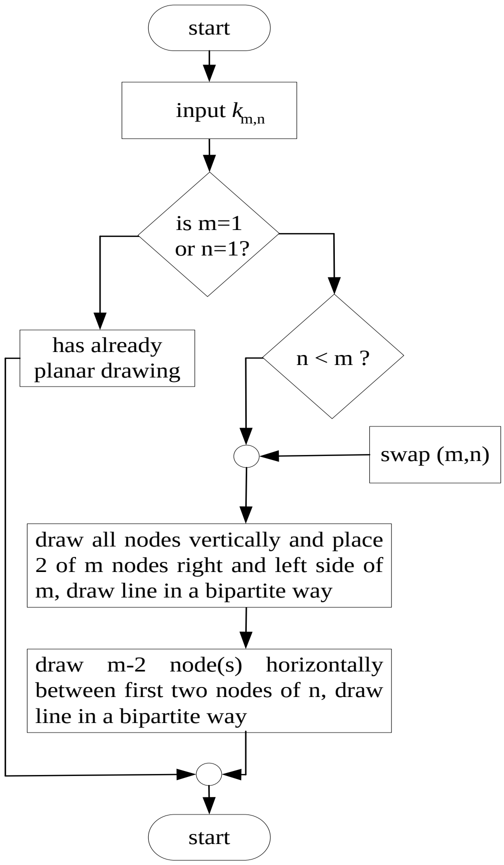

Fig 7.Flow Chart of Planar drawing of Bipartite

VII. Conclusion

A general formula for obtaining the maximum number of edges in the planar drawings of bipartite graphs has been derived. Some important observations and characteristics analysis have been provided.

A general formula that states how many edges must be eliminated to obtain the planar drawings of bipartite graphs has been derived. A constructive proof has been provided and from the proof an algorithm for finding the planar drawings of bipartite graphs has been developed.

From the developed algorithm planar drawings of any n-partite graphs can be developed and an algorithm for finding the planar drawings of any bipartite graph can be derived in the future research work.

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